### None of the organisms in this world can survive alone. To survive in this world, all organisms and their populations interact with one another and with the environment. This population interaction is generally between two different species populations. These interactions can be beneficial or neutral or detrimental. There are six types of population interaction such as Predation, Competition, Mutualism, Parasitism, **Commensalism, Amensalism. Here we will discuss about Predation, Competition and Mutualism.**

### ****Predation/Prey-predator situation:** If the growth rate of one population is decreased and the other is increased, then the populations are in a prey- predator situation.**

### ****Competition situation:** If the growth rate of each population is decreased, then the populations are in a competition situation.**

### ****Mutualism/ Symbiosis situation:** If the growth rate of each population is increased, then the populations are in a mutualism or symbiosis situation.**

**Two species population model:** Let there are two interacting and competing species living in the same environment. If  and  are populations of these species, then the following system



where and  are linear or nonlinear in the variables  and , is called a two species population model. Two species population models are two types such as

(a) Linear population model of two species and

(b) Non-linear population model of two species.

Example: (1). ,  is a linear population model of two species.

(2). ,  is a non-linear population model of two species.

**Non-linear population model of two species:** There are three well known non-linear population models of two species such as

1. **Lotka-Volterra Predator-Prey model:** If the growth rate of one population is decreased and the other

is increased, then the populations are in predator-prey situation (e.g., the populations of rats and cats or of rabbits and foxes etc.). Let  and  denote the populations of prey and predator respectively. Then the Lotka-Volterra predator-prey model is defined as



where , ,  and are positive constants.

1. **Lotka-Volterra Competitive model:** If the growth rate of each population is decreased then the

populations are in competitive situation. Here two species compete for the same limited food source or in some way inhibit each other’s growth. For example, competition may be for territory which is directly related to food resources. When two species compete for the same limited resources one of the species usually becomes extinct.Let  and  denote the competing species at any time . Then the Lotka-Volterra competitive model is defined as



where , ,  and are positive constants.

1. **Lotka-Volterra mutualism or symbiosis model:** If the growth rate of each population is enhanced

then the populations are in mutualism or symbiosis situation. There are many examples where the interaction of two or more species is to the advantage of all. For example, flowering plants are being pollinated by animals, vascular plants are being dispersed by animals.

Let  and  denote the symbiotic species at any time . Then the Lotka-Volterra mutualism or symbiosis model is defined as



where , ,  and are positive constants.

**Lotka-Volterra Prey-Predator model:** In this model, the growth rate of one population is decreased and the other is increased. If  and  denote the populations of prey and predator respectively at time , then the Lotka-Volterra prey -predator model is defined as

 

where , ,  and are positive constants,  and  are the specific growth rates of  and respectively,  and  are the coefficients of predation. The first equation of (1) is called the prey equation and the second equation of (1) is called the predator equation.

**Explanation:** From the prey equation, it is observed that in the absence of predator, the prey population increases and in the presence of predator, the prey population decreases. Similarly, from the predator equation, it is observed that in the absence of prey, the predator population decreases and in the presence of prey, the predator population increases. The contacts between prey and predator are harmful for the prey and are useful for the predator. If the prey and the predator do not interact with each other, then their number will be changed according to the following assumptions:

1. When the predator is absent, then the model becomes,





where . This shows that  as  i.e. when the predator is absent then the prey grows exponentially.

1. When the prey is absent, then the model becomes,





where . This shows that  as  i.e. when the prey is absent then the predator decays exponentially.

**Stability analysis:** The system (1) cannot be solved explicitly and so the exact paths of (1) cannot be drawn in the plane. However we can find equilibrium points where the population remains constant.

For equilibrium or critical points putting  and  in (1), we have

 

Solving (2) we find the equilibrium points  and . We now investigate the stability of these equilibrium points.

**Case-I: Equilibrium point:** This is the case where both the species are absent. Thus when the

populations are sufficiently close to this points, we can neglect the second terms  and  of (1) in comparison to the first terms. Hence (1) reduces to

 

The solutions of (3) are

 

where and  are arbitrary constants. Since  and  are non-negative so we must have  and and the family of curves described by (4) is shown in figure -1.

Figure-1

Further, if we displace the population slightly from the equilibrium point ****, then it tends to move away from the point **.** Thus the equilibrium point ****is unstable.

**Case-II: Equilibrium point :** This is the case where both the species are present. We linearize (1) by using

 

This transforms the point ****to the origin ****and makes ****and  small. Putting (5) in (1) we obtain,

 

Since  and  are small so we can neglect the terms of  and then we obtain

 

The characteristic equation of (7) is,







Hence ****is a stable centre or neutral.

Again from (7), we have





Integrating, 



where  is an integrating constant. This represents a family of concentric ellipses. We find from (7) that the directions of these ellipses are anti-clockwise. Hence the stable centre****is shown in figure-2.

Figure-2

**Exact/ implicit solution:** From (1), we have







Integrating, 







 

where  is an integrating constant.

This is the general solution of (1).

Using initial populations and , we get



Putting this value in (8), we get

 

This is the particular solution of (1).

Now let  and . Then from (8), we get

 

Looking carefully at , we note that

1.  when 
2.  when 

Also 

and 

Thus the critical value of  is  and  is  at .

Hence  has a local maximum at . Thus  looks like the following figure-3

Figure-3

Obviously, will have a similar curve. Since the product of  and is a constant so  looks like of the following figure-4

Figure-4

Figure-5

This implies that the path of (1) is a closed loop as shown in figure-5. Hence from figures 1, 2 and 5 the population dynamics of (1) near its equilibrium points are shown in the following phase plane.

Figure-6

**Lotka-Volterra Competitive model:** In this model, the growth rate of each population is decreased.Here two species compete for the same limited food source or in some way inhibit each other’s growth. If  and  denote the competing species at any time , then the Lotka-Volterra competitive model is defined as

 

where , ,  and are positive constants,  and  are the specific growth rates of  and respectively,  and  are the coefficients of competition.

**Explanation:** From the 1st equation, it is observed that in the absence of , the species  increases and in the presence of , the species  decreases. Similarly, from the 2nd equation, it is observed that in the absence of , the species  increases and in the presence of , the species  decreases. The contacts between the species have an inhibiting effect upon the growth of the species. If the species do not interact with each other, then their number will be changed according to the following assumptions:

1. When the species  is absent, then the model becomes,





where . This shows that  as  i.e. when the species  is absent then another species  grows exponentially.

1. When the species  is absent, then the model becomes,





where . This shows that  as  i.e. when the species  is absent then another species also grows exponentially.

**Stability Analysis:** The system (1) cannot be solved explicitly and so the exact paths of (1) cannot be drawn in the plane. However we can find equilibrium points where the population remains constant.

For equilibrium or critical points putting  and  in (1), we have

 

Solving (2) we find the equilibrium points  and . We now investigate the stability of these equilibrium points.

**Case-I: Equilibrium point :** This is the case where both the species are absent. Thus when the

populations are sufficiently close to this points, we can neglect the second terms  and  of (1) in comparison to the first terms. Hence (1) reduces to

 

The solutions of (3) are

 

where and  are arbitrary constants. Since  and  are non-negative so we must have  and and the family of curves described by (4) is shown in figure -1.

Figure-1

Further, if we displace the population slightly from the equilibrium point ****, then it tends to move away from **.** Thus the equilibrium point ****is unstable.

**Case-II: Equilibrium point :** This is the case where both the species are present. We linearize (1) by using

 

This transforms the point ****to the origin ****and makes ****and  small. Putting (5) in (1) we obtain,

 

Since  and  are small so we can neglect the terms of  and then we obtain

 

The characteristic equation of (7) is,







The roots are real, unequal and of opposite signs.

Hence ****is an unstable saddle.

Again from (7), we have





Integrating, 

 

where is an integrating constant. This represents a family of hyperbolas. The unstable saddle ****is shown in figure-2.

Figure-2

**Exact/ implicit solution:** From (1), we have







Integrating,  

where is an integrating constant.

This is the general solution of (1).

Using initial populations and , we get



Putting this value in (8), we get



 

This is the particular solution of (1).

To find the equation of , we note that the left side of (9) has a maximum at  and the right side of (9) has a maximum at . The separatrix occurs when the initial values have the critical value ****. Thus is given by



 

The equation (11) represents the curve  for **** or  for **.**.

Figure-3

**Lotka-Volterra Symbiosis model:** In this model, the growth rate of each population is enhanced. If  and  denote the symbiotic species at any time , then the Lotka-Volterra mutualism or symbiosis model is defined as

 

where , ,  and are positive constants,  and  are the specific growth rates of  and respectively,  and  are the coefficients of cooperation.

**Explanation:** From the first equation of (1) it is observed that in the absence of , the species decreases exponentially and in the presence of , the species  increases. Similarly, from the second equation of (1) it is observed that in the absence of , the species  decreases exponentially and in the presence of , the species  increases. Thus the presence of each species has a cooperative effect upon the growth of the other species. If the species do not interact with each other, then their number will be changed according to the following assumptions:

1. When the species  is absent, then the model becomes,





where . This shows that  as  i.e. when the species  is absent then another species  decays exponentially.

1. When the species  is absent, then the model becomes,





where . This shows that  as  i.e. when the species  is absent then another species also decays exponentially.

**Stability analysis:** The system (1) cannot be solved explicitly and so the exact paths of (1) cannot be drawn in the plane. However we can find equilibrium points where the population remains constant.

For equilibrium or critical points putting  and  in (1), we have

 

Solving (2) we find the equilibrium points and . We now investigate the stability of these equilibrium points.

**Case-I: Equilibrium point :** This is the case where both the species are absent. Thus when the

populations are sufficiently close to this points, we can neglect the second terms  and  of (1) in comparison to the first terms. Hence (1) reduces to

 

The solutions of (3) are

 

where and  are arbitrary constants. Since  and  are non-negative so we must have  and and the family of curves described by (4) is shown in figure -1.

Figure-1

Further, if we displace the population slightly from the equilibrium point ****, then it tends to move towards and enters the equilibrium point **.** Thus the equilibrium point ****is asymptotically stable.

**Case-II: Equilibrium point :** This is the case where both the species are present. We linearize (1) by using

 

This transforms the point ****to the origin ****and makes ****and  small. Putting (5) in (1) we obtain,

 

Since  and  are small so we can neglect the terms of  and then we obtain

 

The characteristic equation of (7) is,







The roots are real, unequal and of opposite signs.

Hence ****is an unstable saddle.

Again from (7), we have





Integrating, 

 

where is an integrating constant. This represents a family of hyperbolas. Hence from (8) and figure-1, the saddle ****is shown in figure-2.

Figure-2

**Exact/ implicit solution:** From (1), we have







Integrating,  

where is an integrating constant.

This is the general solution of (1).

Using initial populations and , we get



Putting this value in (9), we get







 

This is the particular solution of (1).

To find the equation of , we note that the left side of (10) has a maximum at  and the right side of (10) has a maximum at . The separatrix occurs when the initial values have the critical value ****. Thus is given by



 

The equation (11) represents the curve  for ****or for **.**

Figure-3

**Questions**

**Q-01:** Discuss two species Lotka-Volterra Prey-Predator model and find its exact solution. Make a stability analysis of this model.

**Q-02:** Discuss two species Lotka-Volterra competitive model and find its exact solution. Make a stability analysis of this model.

**Q-03:** Discuss two species Lotka-Volterra mutualism or symbiosis model and find its exact solution. Make a stability analysis of this model.

**Q-04:** Write down the two species Lotka-Volterra competitive model. Find all the equilibrium points and discuss the nature of stability at any which has both species present.

**Q-05:** Describe the two species Lotka-Volterra competitive model and show that the trajectory in the neighborhood of an equilibrium point is a hyperbola.

**Q-05:** Describe the two species Lotka-Volterra Prey-Predator model and show that the trajectory in the neighborhood of an equilibrium point is an ellipse.

**Q-06:** Discuss Lotka-Volterra model for two species interaction of Prey-Predator type.

**Problems**

**Problem-01:** Discuss the solutions of the system

****

**Solution:** The given system is

**** 

For equilibrium or critical points we know,  and .i.e.

 

 

Solving (2) and (3) we get the critical points  and .

We now examine the local behavior of solutions near each critical point.

**Case-01:** For the critical point , putting the transformation , , where  and  in (1) , we get

****

The corresponding linear system is,

**** 

If be the eigen value then the characteristic equation of (4) is,

****

****

****

Since the values of are opposite in sign so the origin is a saddle point of both the linear system (4) and of the nonlinear system (1), and therefore is unstable.

Let  be an eigen vector of (4), then

**** 

When , then from (5), we have

****

****

****

Let ****. The eigen vector is .

Again, when , then from (5), we have

****

****

****

Let ****. The eigen vector is .

Therefore the general solution is,

****.

**Case-02:** For the critical point , putting the transformation , , where  and  in (1) , we get

****

****

The corresponding linear system is

** **

If be the eigen value then the characteristic equation of (6) is,

****

****

****

Since the values of are imaginary, the critical point  is a centre of the linear system (6) and is therefore a stable critical point for that system.

Let  be an eigen vector of (6), then

**** 

When , then from (7), we have

****

****

**** 

Solving the equations of (8), we get

 and .

The eigen vector is .

Again, When , then from (7), we have

****

****

**** 

Solving the equations of (9), we get

 and .

The eigen vector is .

Therefore the general solution is,

****.

**Problem-02:** Discuss the qualitative behavior of solutions of the system

****.

**Solution:** The given system is

**** 

For equilibrium or critical points putting  and  in (1), we get

 

**** 

From (2),  and  

From (3),  and  

Solving (4) and (5), we get

, 

From (5), putting , we get 

From (4), putting , we get 

Therefore the critical points or equilibrium points are ,, and . They correspond the equilibrium solutions of the system (1). The first three of these points involve the extinction of one or both species, only the last corresponds to the long term survival of both species.

We now discuss the stability of the solutions near each critical point. The system (1) is almost linear in the neighbourhood of each critical point.

**Case-01:** For the critical point , putting the transformation  ,  in (1) , we get

****

The corresponding linear system is,

**** 

If be the eigen value then the characteristic equation of (6) is,

****

****

****

Let  be an eigen vector of (6), then

**** 

When , then from (7), we have

****

****

****

Let ****. The eigen vector is .

Again, when , then from (7), we have

****

****

****

Let ****. The eigen vector is .

Therefore the general solution is,

****.

Since the values of are positive, so the solution is unstable.

**Case-02:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

**** 

If be the eigen value then the characteristic equation of (8) is,

****

****

****

Let  be an eigen vector of (8), then

**** 

When , then from (9), we have

****

****

Let ****. The eigen vector is .

Again, when , then from (9), we have

****

****

Let ****then **** . The eigen vector is .

Therefore the general solution is,

****.

Since the values of are opposite signs, the point  is a saddle point. Hence  is an unstable equilibrium point of the linear system (8) and of the nonlinear system (1).

**Case-03:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

**** 

If be the eigen value then the characteristic equation of (10) is,

****

****

****

Let  be an eigen vector of (10), then

**** 

When , then from (11), we have

****

****

Let **** then ****. The eigen vector is .

Again, when , then from (11), we have

****

****

Let ****. The eigen vector is .

Therefore the general solution is,

****.

Since the values of are opposite signs, the point  is a saddle point. Hence  is an unstable equilibrium point of the linear system (10) and of the nonlinear system (1).

**Case-04:** For the critical point , putting the transformation  , in (1), we get

****

****

The corresponding linear system is

**** 

If be the eigen value then the characteristic equation of (12) is,

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****

Let  be an eigen vector of (12), then

**** 

When , then from (13), we have

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Let **** then ****. The eigen vector is .

Again, When , then from (13), we have

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****

**** 

Let **** then ****. The eigen vector is .

Therefore the general solution is,

****.

Since the values of are negative, the point  is an asymptotically stable of the linear system (12) and of the nonlinear system (1).

**Problem-03:** Find the equilibrium points of

****

and draw some trajectories.

**Solution:** The given system is

**** 

For equilibrium or critical points we know,  and .i.e.

 

 

Solving (2) and (3) we get the critical points  and .

We now examine the local behavior of solutions near each critical point.

**Case-01:** For the critical point , putting the transformation ,  in (1) , we get

****

The corresponding linear system is,

**** 

If be the eigen value then the characteristic equation of (4) is,

****

****

****

Since the values of are positive, so the origin is unstable.

**Case-02:** For the critical point , putting the transformation ,  in (1) , we get

****

****

The corresponding linear system is

** **

If be the eigen value then the characteristic equation of (5) is,

****

****

****

Since the values of are opposite in signs, so the critical point  is a saddle point. Hence is an unstable point of the linear system (5) and of the nonlinear given system (1).

From (5), we have

****

****

Integrating, ****

****.

This represents a family of hyperbolas. Thus the trajectories in the neighborhood of the equilibrium point are hyperbola. The line  and  divide the first quadrant into four parts.

Figure-01: page 151 of problem -09

**Problem-04:** Find the equilibrium points of

****

and sketch some trajectories.

**Solution:** The given system is

**** 

For equilibrium or critical points we know,  and .i.e.

 

 

Solving (2) and (3) we get the critical points  and .

We now examine the local behavior of solutions near each critical point.

**Case-01:** For the critical point , putting the transformation ,  in (1) , we get

****

The corresponding linear system is,

**** 

If be the eigen value then the characteristic equation of (4) is,

****

****

****

Since the values of are positive, so the origin is unstable.

**Case-02:** For the critical point , putting the transformation ,  in (1) , we get

****

****

The corresponding linear system is

** **

If be the eigen value then the characteristic equation of (5) is,

****

****

****

Since the values of are opposite in signs, so the critical point  is a saddle point. Hence is an unstable point of the linear system (5) and of the nonlinear given system (1).

From (5), we have

****

****

Integrating, ****

****.

This represents a family of hyperbolas. Thus the trajectories in the neighborhood of the equilibrium point are hyperbola. The line  and  divide the first quadrant into four parts.

Figure-01: page 153 of problem -10